Hopf bifurcation and quasiperiodicy in a simulation model of the leaky faucet

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Simulations with a discrete mapping, obtained from a relaxation oscillator model for the leaky faucet, are presented. It is shown that Hopf bifurcations and closed-loop attractors similar to those observed in leaky faucet experiments can be easily reproduced. [S1063-651X(98)01111-8]

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In dripping faucet experiments complex dynamical behavior, including chaos, has been observed [1–8]. Among other things, quasiperiodicity [3] and, recently, the occurrence of a Hopf bifurcation was experimentally detected [7]. In a previous paper [8], analog simulations were presented of a relaxation oscillator model, which includes substantial improvements over the variable mass oscillator model of Shaw [1]. These improvements allow for the experimental observation that the forming drop undergoes a stretching during the breaking-off and thus emphasize the importance of the discontinuity introduced at the critical point. The results of the numerical investigations showed evidence of the capability of this model to reproduce the broad variety of phenomena experimentally observed [8,9].

As remarked by Bernhard [10], the nonlinearities that are essential for aperiodic motion in a chaotic relaxation oscillator are the discontinuities in the dynamical equations. On the basis of this observation, a discrete mapping technique was proposed to represent the dynamics of a leaky tap [11]. With this mapping the whole scenario of dynamical behavior of the dripping faucet can be reproduced, greatly reducing the computational time.

In the present paper it is shown that, among other features, the mapping also reproduces the quasiperiodical behaviors seen in the real leaky tap [3,7]. The differential equations of the variable mass oscillator model [8] can be put, in a straightforward way, in the following dimensionless form:

$$\frac{dX}{dT}=V,$$

$$M\,\frac{dV}{dT}=-KX-V+M,$$

$$\frac{dM}{dT}=F, \tag{1}$$

where T is the time normalized to $\tau=(x_c/g)^{1/2}$, g is the gravitational acceleration, $X=x/x_c$ is the displacement normalized to the critical displacement x_c, M is the mass normalized to $\mu=b\tau$, b is the coefficient of friction, K is a spring constant normalized to b/τ, and F represents the flow rate normalized to b.

The mechanism allowing for the drop detachment is simulated by reducing the mass by an amount

$$\Delta M=\alpha MV \tag{2}$$

(where α is a constant) at the critical point (here $X_c=1$).

A stretching of the forming drop of mass M is supposed at the threshold, so that, after the falling off of the drop, the residual restarts with velocity V_c at the point

$$X_0=1-R\,\frac{\Delta M}{M}, \tag{3}$$

where

$$R=\left(\frac{3\Delta M}{4\pi D}\right)^{1/3} \tag{4}$$

represents the drop radius and D the liquid density normalized to μ/x_c^3.

A solution of the set of Eq. (1) can be approximated by the function [11]

$$X(T)=(A\sin\Omega T+B\cos\Omega T)e^{-T/M}+M/K, \tag{5}$$

where

$$\Omega=\sqrt{K/M}. \tag{6}$$

The mapping is obtained by finding, for each drop, the values of T that satisfy the equation

$$X(T)=1. \tag{7}$$

At fixed values of the parameters K and α bifurcation diagrams are obtained by calculating the time intervals T_n between successive drops as a function of the flow rate F, by using Eq. (5) and conditions (2) and (3) [12].

Figure 1 shows a particular region of F values where an evolution from a periodic to a quasiperiodic behavior is produced. This transition is characterized by the emerging, from a period-1 region, of a closed-loop pattern that further evolves towards a strange attractor. Starting from about $F=0.578$ up to the period-7 region, return maps T_{n+1} versus

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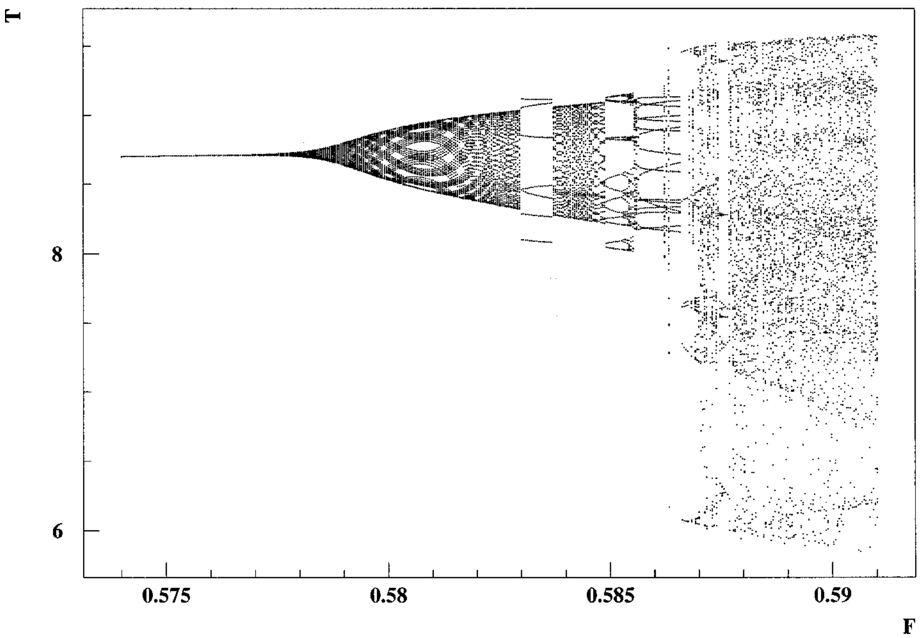

FIG. 1. A selected region of a bifurcation diagram for the relaxation oscillator model for the dripping faucet as F is varied, at $\alpha=8$ and $K=15$. The spectrum consists of 50 points for each F value.

T_n show that a spiral node evolves to a quasiperiodic motion, by opening a limit circle of small radius and giving rise to limit circles with increasing size as the control parameter F increases; then the T^2 torus evolves to a period-7 frequency locking. The power spectra of the corresponding pseudotemporal series display the appearance of various linear combinations of two base frequencies. Thus the pseudotemporal series undergoes a sequence of transitions very similar to those described in the papers quoted in Ref. [7] for the actual dripping faucet, an evident difference being that the experimental evolution occurs from a period-5 frequency to a period-1 motion through a quasiperiodic movement, an inverse transition with respect to that shown in Fig. 1, where the quasiperiodic movement leads to a period-7 behavior.

In Fig. 2 a closed loop and a chaotic attractor are shown. By looking in the region of the map of Fig. 2(b) around $T_n \sim 8.7$, one can see that the strange attractor develops by means of successive distortions of the loop, followed by elongation of corners, which represents a stretching apart of nearby points [9].

The birth of a stable limit cycle pictured in Fig. 1 is a (secondary) Hopf bifurcation and can be modeled [13] using the two-dimensional map $(r,\theta)\rightarrow(r',\theta')$,

$$\begin{aligned} r' &= (F-F_0+1)r + ar^3, \\ \theta' &= \theta + c. \end{aligned} \tag{8}$$

Let us assume $a<0$. Then, if $F>F_0$, an invariant circle is defined, the radius of which is given by

$$r_0^2 = -\frac{1}{a}(F-F_0), \tag{9}$$

where F_0 is the critical control parameter. The radius r_0 can be chosen as the mean distance of the points (T_{n+1},T_n) on the limit circle from their center $(\bar{T},\bar{T})$, where

$$\bar{T} = \frac{1}{N}\sum_{n=1}^{N} T_n$$

is the mean dripping time. From a linear fit, prescribed by Eq. (9), we obtain $F_0=0.5790$.

In Fig. 3 a double Hopf bifurcation obtained at $\alpha=4.4$ and $K=37$ is shown. Increasing F, after a period doubling bifurcation of a period-1 behavior, two spiral nodes form

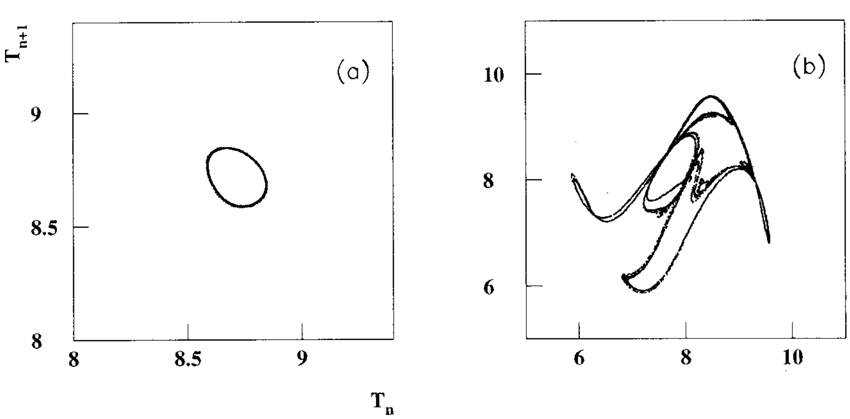

FIG. 2. Return maps T_{n+1} vs T_n, showing (a) a closed loop pattern at $F=0.5795$ and (b) a strange attractor at $F=0.5900$. For each map 2^{14} points are used.

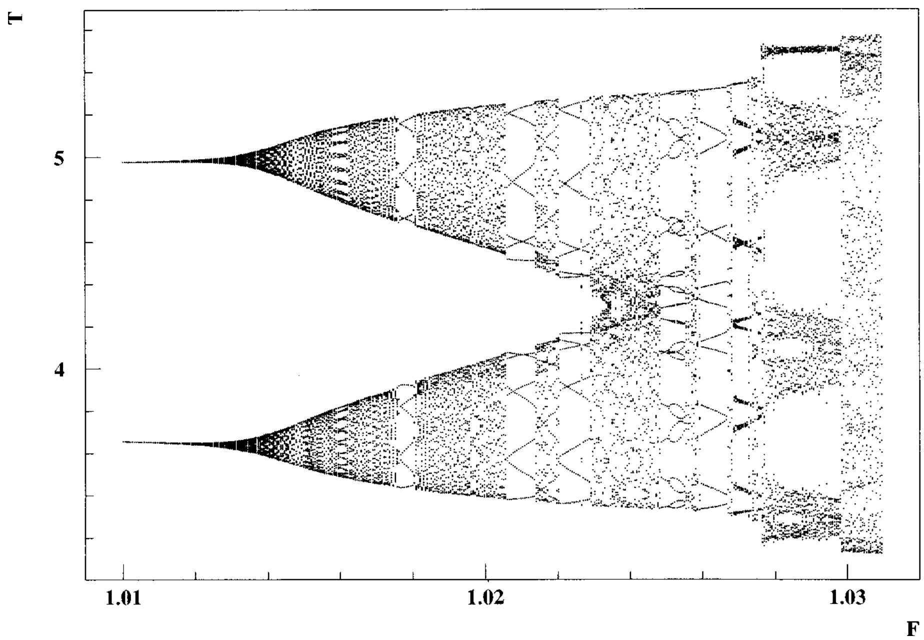

FIG. 3. Birth of a double Hopf bifurcation from a period-2 behavior. Parameters values are $\alpha = 4.4$ and $K = 37$.

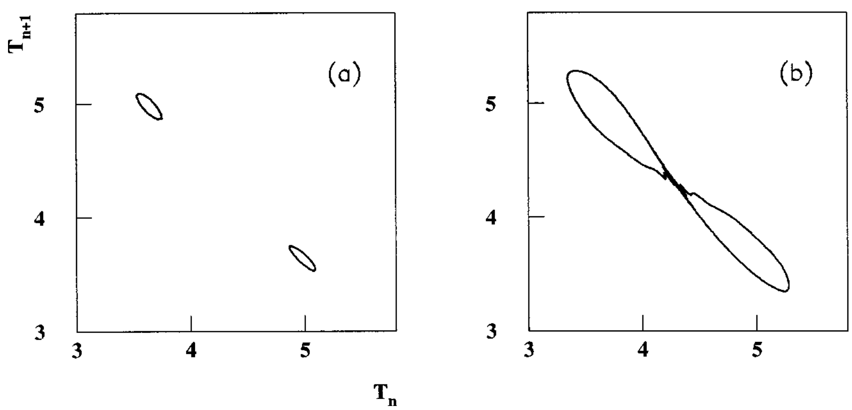

FIG. 4. Return maps at (a) $F = 1.0150$ and (b) $F = 1.0235$.

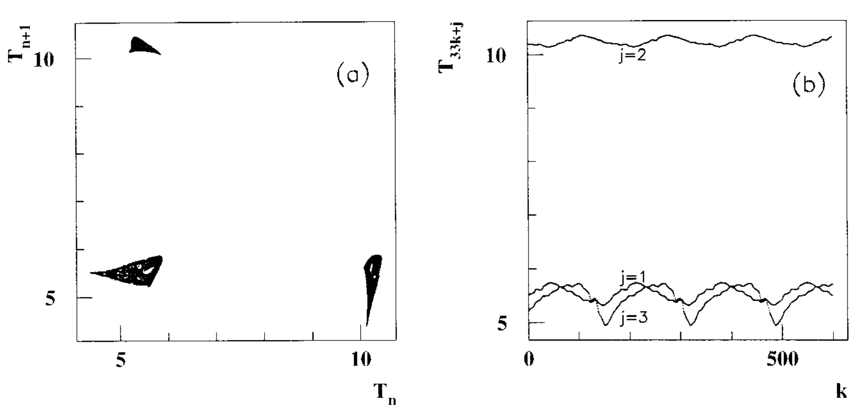

FIG. 5. (a) Chaotic triple closed loop at $F = 0.3600$, $\alpha = 5$, and $K = 35$. (b) Time series at $F = 0.3620$; plots of every 33rd T_n, starting from T_1, T_2, and T_3.

stable limit cycles as those of Fig. 4(a); then, proceeding across alternate regions of periodic and quasiperiodic patterns, the loops increase their sizes, evolve to larger amplitude, and strike and twist (at $F \sim 1.023$) by forming a bow as in Fig. 4(b). Increasing again the flow rate, the knot of the bow unties and the attractor acquires a loop form, which afterwards coalesces in points giving rise to period-7 frequency locking motion. At $F \sim 1.0305$, a sort of loop with complex behavior develops.

Finally, triple closed loops very similar in aspect to the experimental ones reported in Fig. 8(b) of Ref. [3] are obtained at $\alpha = 5$ and $K = 35$ around $F \sim 0.36$. Varying the flow rate, chaotic triple loops as those shown in Fig. 5(a), multiperiodicity (e.g., at $F = 0.3610$) and regular triple loops (e.g., at $F = 0.3620$) are found. The time series diagrams for this last triple-loop pattern are shown in Fig. 5(b), where T_{33k+j} $(j = 1,2,3)$ versus k are plotted. Each curve corresponds to one of the loops and indicates quasiperiodicity. An analogue separation of experimental time series of Ref. [3] into three patterns occurs by plotting T_{15k+j}.

In conclusion, we have shown that the mapping of a relaxation oscillator analog to the dripping faucet enables us to obtain chaotic attractors very similar to the experimental ones, and in particular to simulate Hopf bifurcations and reproduce multiple closed loops. The variety of dynamical behavior reported in the present and early papers [8,9,11] yields good reasons to believe that the model is adequate to shed light on the physical mechanism of the dripping faucet dynamics. It seems clear that the mechanism that permits the modeling of the complex leaky tap physics is the swift change at the threshold. Equations (3), (4), and (5) can provide some suggestions to improve the model proposed. For example, the dynamics greatly changes if one permits the density D to vary, since this parameter controls the rebound.

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